An exterior for the Gödel spacetime

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Abstract

We match the vacuum, stationary, cylindrically symmetric solution of Einstein's field equations with Λ, in a form recently given by Santos, as an exterior to an infinite cylinder of dust cut out of a Gödel universe. There are three cases, depending on the radius of the cylinder. Closed timelike curves are present in the exteriors of some of the solutions. There is a considerable similarity between the spacetimes investigated here and those of van Stockum referring to an infinite cylinder of rotating dust matched to vacuum, with $\Lambda = 0$.

1 Introduction

van Stockum [1] solved the problem of a rigidly rotating infinitely long cylinder of dust, using Einstein's equations without cosmological constant. The metric for the interior is unique, depending on one parameter a; but the vacuum exterior has three cases, depending on the mass per unit length of the interior. The metric corresponding to the case of lowest mass can be diagonalised locally (but not globally), but this is not possible for the other two cases. (See [2] and references given therein.)

In this paper we consider a similar problem using Einstein's equations with negative cosmological constant. For the interior containing rotating matter we use an infinite cylinder cut out of a Gödel universe. The exterior metric has been given by various authors, notably Krasiński [4] (see [9] for more information). However, in this work we use it in the form recently obtained by Santos [3]; it satisfies

$$R^{ik} - \frac{1}{2}g^{ik}R = \Lambda g^{ik}, \tag{1}$$

It turns out that in this problem too the exterior has three cases, depending on the radius (and therefore mass per unit length) of the cylinder.

2 The metrics

For the interior we use the Gödel metric in the form

$$ds^2 = dR^2 + dZ^2 + 4b^2(\sinh^2\rho - \sinh^4\rho)d\psi^2 - 4(2^{\frac{1}{2}})b\sinh^2\rho\, d\psi\, dT - dT^2, \quad (2)$$

where

$$\rho = R/2b, \quad b = (-2\Lambda)^{-1/2}.$$

Throughout the paper positive square roots are to be taken unless the contrary is indicated. The ranges of the coordinates are

$$R \leq R_0, \ -\infty < Z < +\infty, \ 0 \leq \psi \leq 2\pi, \ -\infty < T < +\infty,$$

and $\psi = 0$ and $\psi = 2\pi$ are to be identified.

The exterior metric is that of [3] for $\Lambda < 0$ in slightly different form:

$$ds^2 = dr^2 + \exp\mu\, dz^2 + l d\phi^2 + 2k\, d\phi\, dt - f dt^2, \quad (3)$$

μ, l, k, f being functions of r only. We shall write

$$D^2 = lf + k^2, \ \ G = e^{\mu/2}D, \quad (4)$$

and suppose that D and G are positive. Then the complete solution can be written

$$\begin{aligned}
G &= C_1\cosh\sqrt{3/2}(r/b) + C_2\sinh\sqrt{3/2}(r/b), && (5)\\
e^{3\mu/2} &= \epsilon G\exp(\delta\Theta/\gamma), && (6)\\
l &= -D\alpha^{-1}\sinh(\alpha\Theta/2\beta), && (7)\\
k &= D[\cosh(\alpha\Theta/2\beta) + (\beta/\alpha)\sinh(\alpha\Theta/2\beta)], && (8)\\
f &= D[2\beta\cosh(\alpha\Theta/2\beta) + \alpha^{-1}(\alpha^2+\beta^2)\sinh(\alpha\Theta/2\beta)], && (9)\\
\Theta' &= \gamma/G, && (10)
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, C_1, C_2$ are constants, all real except α which can be real or imaginary; they must satisfy one relation

$$\alpha^2\gamma^2/\beta^2 = 8b^{-2}(C_2^2 - C_1^2) - 4\delta^2/3; \quad (11)$$

a prime, as in (10), means d/dr. Another constant, not occurring in (11), arises in the integration of (10). One can easily check that (7),(8),(9) give real, finite expressions if α is imaginary or is allowed to tend to zero.

The issue of the constants arising in this solution is discussed in another paper and here we shall simply show how they can be fixed by matching the solution to the Gödel metric. The same matching procedure applies whether we take the Gödel metric as interior or exterior. We choose the former because it yields a globally regular solution, whereas with a Gödel exterior and Santos interior it seems that we must allow a singularity along the symmetry axis.

3 Matching the metrics

We wish to match the metrics (2) and (3) across a hypersurface Σ which is $R = R_0$ in (2) and $r = r_0$ in (3) ($R_0, r_0 > 0$). With respect to this hypersurface both the metrics are in Gaussian form, and we assume that on it the coordinates (Z, ψ, T) and (z, ϕ, t) are the same and have the same ranges. Then the Lichnerowicz matching conditions will be satisfied if we equate on Σ the metric components and also their first derivatives. This means that μ, k, l, f and their derivatives need to take the corresponding values from the metric (2). For the time being we exclude the cases in which

$$\sinh^2 \frac{R_0}{2b} = \frac{1}{2}; \;\; \sinh^2 \frac{R_0}{2b} = 1 \tag{12}$$

We shall consider the matching problem in the form

$$\begin{array}{rcll}
 & e^{\mu} & 1 & (13) \\
 & \mu' & 0 & (14) \\
k^2 + fl = & G^2 e^{-\mu} & b^2 \sinh^2 2\rho_0 & (15) \\
f'l - l'f = & G^2 e^{-\mu}\Theta' & -2b \sinh 2\rho_0 (1 - 2\sinh^2 \rho_0) & (16) \\
f'k - k'f = & G^2 e^{-\mu}\Theta'(\alpha^2 - \beta^2)/2\beta & 2^{\frac{1}{2}} \sinh 2\rho_0 & (17) \\
k'l - l'k = & G^2 e^{-\mu}\Theta'/2\beta & -4(2^{\frac{1}{2}}) b^2 \sinh 2\rho_0 \sinh^4 \rho_0 & (18) \\
k/l = & -\alpha \coth(\alpha\Theta/2\beta) - \beta & -[2^{\frac{1}{2}} b (1 - \sinh^2 \rho_0)]^{-1} & (19) \\
f/l = & -2\alpha\beta \coth(\alpha\Theta/2\beta) - (\alpha^2 + \beta^2) & [4b^2(\sinh^2 \rho_0 - \sinh^4 \rho_0)]^{-1} & (20) \\
 & f & 1 & (21)
\end{array}$$

where $\rho_0 := R_0/2b$. The quantities in the right-hand column are obtained from (2), and give the values which must be assumed by the quantities in the left-hand column. It is not hard to show that this set of equalities solves the matching problem as stated in the previous paragraph. Eqn (21) is necessary because (15)-(20) do not determine the sign of the set $[k, l, f, k', l', f']$, so the sign of one member of the set must be prescribed. In this form, which has been chosen for ease of calculation, there is a redundancy: nine equations for eight conditions arising from the continuity of μ, k, l, f and their derivatives.

By a shift of the origin of r (which amounts to a redefinition of C_1, C_2 in (5)) we can arrange that the radial coordinate is continuous on Σ, so $r_0 = R_0$. From the continuity of D and its derivative, and μ and its derivative, we deduce the continuity of G and its derivative. On Σ we have

$$G = b \sinh 2\rho_0, \;\; G' = \cosh 2\rho_0, \tag{22}$$

so, using (5) we find

$$C_1 \cosh \sqrt{6}\rho_0 + C_2 \sinh \sqrt{6}\rho_0 \;\;=\;\; b \sinh 2\rho_0,$$

$$C_1 \sinh \sqrt{6}\rho_0 + C_2 \cosh \sqrt{6}\rho_0 \quad = \quad \sqrt{\frac{2}{3}} b \cosh 2\rho_0,$$

whence

$$C_1 = b[\sinh 2\rho_0 \cosh \sqrt{6}\rho_0 - \sqrt{\frac{2}{3}} \cosh 2\rho_0 \sinh \sqrt{6}\rho_0], \tag{23}$$

$$C_2 = b[\sqrt{\frac{2}{3}} \cosh 2\rho_0 \cosh \sqrt{6}\rho_0 - \sinh 2\rho_0 \sinh \sqrt{6}\rho_0]. \tag{24}$$

We can now write down an expression for G in terms of r, r_0 and b. From (5), (23) and (24) we find, after a short calculation

$$G = b[\sinh 2\rho_0 \cosh \sqrt{\frac{3}{2}} \frac{r - r_0}{b} + \sqrt{\frac{2}{3}} \cosh 2\rho_0 \sinh \sqrt{\frac{3}{2}} \frac{r - r_0}{b}], \tag{25}$$

which shows that G is positive for $r \geq r_0$ as required by (4).

We now proceed with matching (13)-(20), excluding for the time being the two cases (12). The arbitrary constant ϵ can be chosen to satisfy (13), but the continuity of μ' in (14) requires

$$G'_0 + \delta = 0, \tag{26}$$

where the suffix 0 means the value on Σ. Using (22) we find

$$\delta = -\cosh 2\rho_0. \tag{27}$$

(15) is satisfied in virtue of (13) and (22). (16)-(18) lead to

$$(\Theta')_0 \quad = \quad -(b \cosh \rho_0 \sinh \rho_0)^{-1}(1 - 2\sinh^2 \rho_0), \tag{28}$$

$$\beta \quad = \quad (4\sqrt{2} b \sinh^4 \rho_0)^{-1}(1 - 2\sinh^2 \rho_0), \tag{29}$$

$$\alpha^2 \quad = \quad (32 b^2 \sinh^8 \rho_0)^{-1} Y^2, \tag{30}$$

where $Y^2 = 1 - 4\sinh^2 \rho_0 - 4\sinh^4 \rho_0$. The case in which α is zero will be considered in Section 4. If $\alpha \neq 0$ (19) now gives

$$\alpha \coth \frac{\alpha \Theta_0}{2\beta} = -\frac{1 - 3\sinh^2 \rho_0 - 2\sinh^4 \rho_0}{4\sqrt{2} b \sinh^4 \rho_0 (1 - \sinh^2 \rho_0)}, \tag{31}$$

and from (29), (30) and (31) we find that (20) is satisfied.

To verify (21) we first calculate from (31)

$$\alpha^{-1} \sinh \frac{\alpha \Theta_0}{2\beta} = 2b\eta \tanh \rho_0 (1 - \sinh^2 \rho_0), \tag{32}$$

where $\eta = \pm 1$. Using (9) we now write f in the form

$$f = D\alpha^{-1} \sinh \frac{\alpha\Theta}{2\beta}[2\alpha\beta \coth \frac{\alpha\Theta}{2\beta} + \alpha^2 + \beta^2], \tag{33}$$

so that, on the boundary we have after a calculation using (15),(28),(29),(31) and (32)

$$f_0 = -\eta. \tag{34}$$

Therefore to satisfy (21) we must take

$$\eta = -1. \tag{35}$$

Because of (32) and (34) Θ_0 must satisfy

$$\sinh \frac{\alpha\Theta_0}{2\beta} = -2b\alpha \tanh \rho_0 (1 - \sinh^2 \rho_0), \tag{36}$$

where from (30)

$$\alpha = \pm(4\sqrt{2}b \sinh^4 \rho_0)^{-1}(1 - 4\sinh^2 \rho_0 - 4\sinh^4 \rho_0)^{1/2}, \tag{37}$$

both signs on the right being allowed.

Finally we must check that the constants as determined above satisfy (11). First from (10), (22) and (28) we get

$$\gamma = -2(1 - 2\sinh^2 \rho_0), \tag{38}$$

and inserting this, together with the other constants into (11) we find it satisfied.

We have matched Santos's metric (3) at $r = R_0$ *as an exterior to the Gödel metric except for the special cases in which* $\sinh^2 \rho_0$ *is equal to* 1 *or* 1/2. It will now be convenient to consider separately the three cases determined by the sign of α^2.

4 The various cases

We now explore the various cases that can arise, paying special attention to the occurrence of closed timelike curves (CTC). We make use of a theorem of Carter [6,7] (cf. Tipler [8]). It states that in any spacetime with an Abelian isometry group which is everywhere transitive on timelike surfaces, any point can be connected to any other point by both a future and a past timelike curve (and hence a CTC) if and only if there is no Lie algebra covector whose corresponding differential form is everywhere well-behaved and timelike (Carter refers to this form as spacelike or null, but has in mind the nature of its orthogonal hypersurface: we shall not discuss the null case). For the metric (3), the Lie algebra covectors are real-valued linear maps, with constant coefficients, on the space

of vectors of the form $v^t\partial_t + v^\phi\partial_\phi + v^z\partial_z$, so we have to examine the nature of a one-form $\mathbf{w} = Adt + Bd\phi + Cdz$, with constant A, B and C. Such a one-form has $g^{ab}w_a w_b = (-lA^2 + 2kAB + fB^2)/D^2 + C^2 \exp(-\mu)$, and we want to find out whether or not $\mathbf{w}$ can be timelike everywhere, i.e. satisfy $g^{ab}w_a w_b < 0$ for all r. Since the contribution from C is always positive, we need consider only the case $C = 0$. If $l > 0$ for all r, $\mathbf{w} = dt$ is clearly timelike, so there exist no CTC. On the other hand, if $l < 0$ for some r, the circles on which r, z and t are constant are clearly CTC. The outcome is that in our spacetimes, excluding the possibility that $l \geq 0$ but $l = 0$ for some r, there are CTC if and only if $l < 0$ for some r.

4.1 The case $\alpha^2 > 0$

The condition for this is $Y^2 > 0$, i.e.

$$\sinh^2\rho_0 < \frac{1}{2}(\sqrt{2} - 1) \approx 0.207, \tag{39}$$

and if it is fulfilled $\sinh\frac{\alpha\Theta}{2\beta}$ is real and the functions in the solution are hyperbolic with real argument.

We consider the exterior in this case to see whether it contains closed time-like curves, i.e. whether $l < 0$ in $r > r_0$. Because l is continuous on Σ we know that it is positive on $r = r_0$; we shall show that it remains positive for $r > r_0$. We first note from (10),(25) and (38) that $\Theta' < 0$ for $r > r_0$. From this it follows that

$$\frac{d}{dr}(\alpha^{-1}\sinh\frac{\alpha\Theta}{2\beta}) = \frac{\Theta'}{2\beta}\cosh\frac{\alpha\Theta}{2\beta}, \tag{40}$$

is negative for $r > r_0$, because from (29) $\beta > 0$. Hence $\alpha^{-1}\sinh(\frac{\alpha\Theta}{2\beta})$ diminishes from the negative value it has when $\rho = \rho_0$, given by (36). Thus from (7) $l > 0$ in $r > r_0$, and CTC do not exist.

There are no CTC in the Gödel interior either because in (2) $l > 0$ for $R < R_0$ in this case.

4.2 The case $\alpha = 0$

In this case

$$\sinh^2\rho_0 = \frac{1}{2}(\sqrt{2} - 1) \approx 0.207. \tag{41}$$

This value of ρ_0 is critical in that it defines the only radius at which the Gödel model admits circular null geodesics. The functions f, k, l of (3) are given by

$$\begin{aligned} l &= -D\Theta/2\beta, \\ k &= D(2 + \Theta)/2, \\ f &= D(4\beta + \beta\Theta)/2. \end{aligned}$$

β is still given by (29) which yields, when (41) is used,

$$\beta = b^{-1}(1 + \sqrt{2}).$$

It is easy to show that the value of Θ on Σ is negative, namely

$$\Theta_0 = -2(3-\sqrt{2}),$$

and also that

$$\gamma = -2(2-\sqrt{2}),$$

so from (10) $\Theta' < 0$ for $r > r_0$. Now an argument similar to that used in the case $\alpha^2 > 0$ shows that $l > 0$ in $r > r_0$ so there are no CTC in the exterior. There are no CTC in the interior either.

4.3 The case $\alpha^2 = -a^2 < 0$

The hyperbolic functions in (7), (8) and (9) become trigonometric:

$$\begin{aligned} l &= -Da^{-1}\sin(a\Theta/2\beta), && (42)\\ k &= D[\cos(a\Theta/2\beta) + (\beta/a)\sin(a\Theta/2\beta)], && (43)\\ f &= D[2\beta\cos(a\Theta/2\beta) + a^{-1}(\beta^2 - a^2)\sin(a\Theta/2\beta)], && (44) \end{aligned}$$

where a is real, and the constants have to satisfy

$$8b^{-2}(C_2^2 - C_1^2) = 4\delta^2/3 - a^2\gamma^2/\beta^2. \tag{45}$$

To investigate the occurrence of CTC when $\alpha^2 < 0$ we must first integrate (10), G being given by (25). This leads to considerable mathematical complexity, so we shall restrict ourselves to four special cases.

4.3.1 The sub-case $\tanh 2\rho_0 = \sqrt{2/3} \leftrightarrow \sinh^2\rho_0 = \frac{1}{2}(\sqrt{3}-1) \approx 0.366.$

For this value of ρ_0 we have $\sinh^2\rho_0 = \frac{1}{2}(\sqrt{3}-1), Y = \pm i$, and from (29), (37) and (38)

$$\alpha = \pm i(2\sqrt{2}b)^{-1}(2+\sqrt{3}),\ \beta = (2\sqrt{2}b)^{-1},\ \gamma = -2(2-\sqrt{3}). \tag{46}$$

(25) gives

$$G = \sqrt{2}b\exp\sqrt{\frac{3}{2}}\frac{r-r_0}{b}, \tag{47}$$

and (10) yields

$$\Theta = \frac{2}{3}(2\sqrt{3}-3)\exp[-\sqrt{\frac{3}{2}}\frac{r-r_0}{b}] + C, \tag{48}$$

where C is a constant of integration. On $r = r_0$ we have

$$\Theta = \Theta_0 = \frac{2}{3}(2\sqrt{3}-3) + C, \tag{49}$$

and at $r = \infty$

$$\Theta_\infty = C. \tag{50}$$

Our aim is to find whether l, given by (42), which is positive on $r = r_0$, becomes negative as Θ ranges between these two values: if so, this will denote the existence of CTC.

We now insert the above values of $\sinh \rho_0, Y, \alpha, \beta$ into (36) and find after a short calculation

$$\sin[(1 + \frac{1}{2}\sqrt{3})\Theta_0] = -\frac{\sqrt{3}}{2}, \;\; \cos[(1 + \frac{1}{2}\sqrt{3})\Theta_0] = -\frac{1}{2}, \tag{51}$$

whence

$$(1 + \frac{1}{2}\sqrt{3})\Theta_0 = \frac{4\pi}{3} + 2n\pi, \tag{52}$$

where n is a positive or negative integer or zero. We now substitute α and β from (46) and Θ_∞ from (50) into (42) to get the value of l at $r = \infty$:

$$l_\infty = -\frac{2\sqrt{2}bD}{2 + \sqrt{3}} \sin(1 + \frac{1}{2}\sqrt{3})C, \tag{53}$$

so, using (49) and (52) we have

$$l_\infty = -\frac{2\sqrt{2}bD}{2 + \sqrt{3}} \sin[(\frac{4\pi}{3} + 2n\pi) - \frac{\sqrt{3}}{3}]. \tag{54}$$

For any n this sine is positive, so l_∞ is negative and CTC exist in the exterior of the cylinder. They do not exist in the interior for this radius.

4.3.2 The sub-case $\sinh^2 \rho_0 = \frac{1}{2}$

Although α^2 given by (37) is negative in this case, the solution (42)-(44) does not apply because with this special value of $\sinh^2 \rho_0$ it follows from (16) that $\Theta' = 0$ on Σ so from (10) $\gamma = 0$ and Θ is constant throughout the exterior. This leads to a case not included in [3], namely, that in which f and l are proportional. The constant of proportionality is found from (20) to be b^{-2} so we have

$$f = b^{-2}l. \tag{55}$$

Let us introduce a function Φ as in (15), (17) of [3] by

$$\Phi' = v'(b^{-2} + v^2)^{-1}, \;\; v = k/l; \tag{56}$$

then we find

$$f = b^{-1}D\cos b^{-1}\Phi, \; k = D\sin b^{-1}\Phi, \; l = bD\cos b^{-1}\Phi, \tag{57}$$

where

$$\Phi' = \nu G^{-1}, \tag{58}$$

ν being a constant.[1]

[1] (58) follows from (18) of [3], but because of the different r coordinate implied by (1) of [3] and (4) of this paper we must write G instead of D.

The constant ν can be determined from (58) by writing Φ' as

$$\Phi' = \frac{k'l - l'k}{D^2} \tag{59}$$

and substituting on the right hand side from (18), (4) and (22); thus we find

$$\nu = -\sqrt{2}b. \tag{60}$$

The next step is to integrate (58), inserting the value of G obtained from (25) after putting $\sinh \rho_0 = \sqrt{1/2}$, namely

$$G = \frac{b}{\sqrt{3}} \cosh(\sqrt{\frac{3}{2}} \frac{r - r_0}{b} + \zeta)$$

where $\sinh \zeta = 2\sqrt{2}$. Inserting this and (56) into (54) and integrating we get

$$\Phi = -4b \tan^{-1}[\tanh \frac{1}{2}(\sqrt{\frac{3}{2}} \frac{r - r_0}{b} + \zeta)] + E, \tag{61}$$

where E is a constant of integration.

Eqns (57) and (58) are what (7)-(10) become in this special case; of the remaining equations in the complete solution, (5) is unchanged, (6) becomes

$$e^{3\mu/2} = \epsilon G e^{\Phi\delta/\nu}, \tag{62}$$

and in place of (11) we have

$$2(C_2^2 - C_1^2) - \frac{1}{3}\delta^2 b^2 = -\nu^2. \tag{63}$$

One can check that this solution matches the Gödel metric on $r = r_0$ provided $\delta = -2$, and E is given an appropriate value as described below.

We turn to the question whether there are CTC in the exterior in this case. Equating f, k, l in (57) to their values on Σ as given by (2) with $\rho = \rho_0$, and using $\sinh^2 \rho_0 = 1/2$, we find

$$\Phi_0 = b(-\varpi + 2n\pi), \tag{64}$$

where n is an integer and ϖ denotes the principal value of $\cos^{-1}(3)^{-1/2}$, which is 0.955 radians, to three decimal places. From (61)

$$\Phi_0 = -4b \tan^{-1}(\tanh \frac{\zeta}{2}) + E; \tag{65}$$

using $\sinh \zeta = 2\sqrt{2}$ and (64) we get

$$E = \Phi_0 + 4b \tan^{-1}(2)^{-1/2} = b[-\varpi + 4\theta + (4m + 2n)\pi], \tag{66}$$

m denoting an integer and θ the principal value of $\tan^{-1}(2)^{-1/2}$, which is 0.615. On the other hand, putting $r = \infty$ in (61) we obtain

$$\Phi_\infty = -4b\tan^{-1}(1) + E = E - b(4s+1)\pi, \tag{67}$$

where s is an integer. Eliminating E we finally obtain

$$b^{-1}\Phi_\infty = (4\theta - \pi - \varpi) + 2\pi(n + 2m - 2s). \tag{68}$$

l will change sign in $r > r_0$ if $\cos b^{-1}\Phi$ does so. On Σ, $b^{-1}\Phi = b^{-1}\Phi_0$ is in the fourth quadrant, as is clear from (64). However, from (68) $b^{-1}\Phi_\infty$ is in the third quadrant, so $\cos b^{-1}\Phi$ changes sign between $r = r_0$ and $r = r_\infty$ and so therefore does l. Hence CTC must exist in the exterior spacetime, though, as in **4.3.1**, they do not exist in the interior.

4.3.3 The sub-case $\sinh^2\rho_0 = 1$

This is the second exceptional case in (12); it has $l = 0$ on Σ, so the ϕ-coordinate circles are null there. The exterior solution (5)-(11) still applies, and so do the boundary conditions (13)-(21) except for (19) and (20). From (4), (13) and (15) we have $D_0 = 2\sqrt{2}b$ and the continuity of $g_{\phi\phi}$ and $g_{\phi t}$ on Σ require

$$\sinh(\alpha\Theta_0/2\beta) = 0,\ \cosh(\alpha\Theta_0/2\beta) = -1, \tag{69}$$

whence

$$\frac{\alpha\Theta_0}{2\beta} = (2n+1)i\pi, \tag{70}$$

where n is an integer. From the requirement that $g_{tt} = 1$ on Σ we find, using (9) and (70)

$$\beta = -(4\sqrt{2}b)^{-1}. \tag{71}$$

The other constants of the solution are now easily determined from (14)-(18) and (27):

$$\alpha^2 = -7/(32b^2),\ (\Theta')_0 = (\sqrt{2}b)^{-1},\ \delta = -3, \tag{72}$$

and all boundary conditions are satisfied on Σ.

That CTC exist in the exterior can be seen as follows. As $D_0 > 0$ D will be positive in the neighbourhood of Σ, from continuity. Moreover, from (69), (71) and (72) $\alpha^{-1}\sinh(\alpha\Theta_0/2\beta)$ is positive in the exterior neighbourhood of Σ; hence using (42) we see that l decreases from its zero value on Σ and CTC must exist in the exterior.

4.3.4 The sub-case $\sinh\rho_0 > 1$

In this case l is negative within the cylinder, on its boundary, and, by continuity, in the exterior at least near the boundary. Therefore CTC exist inside and outside the cylinder.

5 Conclusion

We have been studying a spacetime satisfying Einstein's equations with negative cosmological constant, describing an infinite cylinder cut from the Gödel universe, surrounded by empty space. The spacetime depends on two parameters, $R_0(= r_0)$, the coordinate radius of the cylinder, and the cosmological constant. Our solution of the matching problem on the boundary of the cylinder shows that the exterior metric is of three different types, depending on the radius of the cylinder. For smaller radii there are no closed timelike curves (CTC) inside the cylinder or in the empty exterior. We have not given a complete treatment of CTC for cylinders with larger radii, but we have shown that for some radii they exist outside the cylinder even though there is none inside. For sufficiently large radii ($\sinh \rho_0 > 1$) CTC exist in the interior and the exterior.

There are similarities between our results and the corresponding ones for the van Stockum spacetimes, which refer to infinite dust cylinders in vacuum with $\Lambda = 0$. These spacetimes also depend on two parameters, and three exterior cases exist, depending on σ, the mass per unit length of the cylinder; moreover, the spacetimes contain CTC in the case of large σ [2].

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7 References

[1] van Stockum W J 1937 *Proc. Roy. Soc. Edin.* **57** 135
[2] Bonnor W B 1980 *J. Phys. A: Math. Gen.* **13** 2121
[3] Santos N O 1993 *Class. Quantum Grav.* **10** 2401
[4] Krasiński A 1975 *Acta Phys. Polon. B* **6** 223
[5] MacCallum M A H 1997 To be published in *Gen. Rel. Grav.*
[6] Carter B 1968 *Phys. Rev.* **174** 1559
[7] Carter B 1978 *Gen. Rel. Grav.* **9** 437
[8] Tipler F J 1974 *Phys. Rev. D* **9** 2203
[9] MacCallum M A H and Santos N O 1997 Submitted to *Class. Quantum Grav.*